

ON A GENERALIZATION OF THE SEATING COUPLES PROBLEM

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ABSTRACT. We prove a conjecture of Adamaszek generalizing the seating couples problem to the case of $2n$ seats. Concretely, we prove that given a positive integer n and $d_1, \dots, d_n \in (\mathbb{Z}/2n)^*$ we can partition $\mathbb{Z}/2n$ into n pairs with differences d_1, \dots, d_n .

1. INTRODUCTION

Preissmann and Mischler [6] proved the following, confirming a conjecture of R. Bacher.

Theorem 1.1. *Let $p = 2n + 1$ be an odd prime. Suppose we are given n elements $d_1, \dots, d_n \in (\mathbb{Z}/p)^*$. Then there exists a partition of $\mathbb{Z}/p - \{0\}$ into pairs with differences d_1, \dots, d_n .*

A simpler proof of this theorem can be found in [4]. Karasev and Petrov, independently, gave a proof of this result along the same lines and provided further generalizations in [3]. In this work, they also conjectured two generalizations of Theorem 1.1, replacing p by an arbitrary integer N . The conjecture in the case that N is even is originally due to Adamaszek.

Conjecture 1.2 ([3, Conjecture 1]). *Let $N = 2n + 1$ be a positive integer. Suppose we are given n elements $d_1, \dots, d_n \in (\mathbb{Z}/N)^*$. Then there exists a partition of $\mathbb{Z}/N - \{0\}$ into pairs with differences d_1, \dots, d_n .*

We will prove the conjecture when N is even:

Theorem 2.4 ([3, Conjecture 2]). *Let $N = 2n$ be a positive integer. Suppose we are given n elements $d_1, d_2, \dots, d_n \in (\mathbb{Z}/N)^*$. Then there exists a partition of (\mathbb{Z}/N) into pairs with differences d_1, d_2, \dots, d_n .*

While finishing this paper we found out that, in his master's thesis [5], T.R. Mezei suggests a possible way to solve the conjecture that is similar to ours. Furthermore, he shows that Theorem 2.4 holds whenever $N = 2p$ for p a prime number.

2. THE EVEN CASE

We recall the following version of the Cauchy-Davenport theorem.

Theorem 2.1 ([1, 1.4]). *If A and B are nonempty subsets of \mathbb{Z}/N where $0 \in B$, and $\gcd(b, N) = 1$ for all $b \in B \setminus \{0\}$, then*

$$|A + B| \geq \min\{N, |A| + |B| - 1\}.$$

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Suppose that we have a partition as in Theorem 2.4. Since the d_i are odd numbers, each pair contains exactly one even number. Therefore, if Theorem 2.4 holds there exists signs s_i such that $s_1 d_1 + \dots + s_n d_n \equiv 1 - 2 + 3 - \dots + (2n - 1) - 2n \equiv n \pmod{N}$.

Theorem 2.2. *Let $N = 2n$ and let $d_1, \dots, d_n \in (\mathbb{Z}/N)^*$. Then there exists $s_1, \dots, s_n \in \{1, -1\}$ such that*

$$s_1 d_1 + \dots + s_n d_n \equiv n \pmod{2n}$$

Proof. It is enough to prove that there exists $I \subset \{1, \dots, n\}$ such that

$$\sum_{i \in I} 2d_i \equiv d_1 + d_2 + \dots + d_n + n \pmod{2n}$$

Since d_i is odd for every i , $d_1 + d_2 + \dots + d_n + n$ is even and therefore our task is equivalent to finding I such that

$$\sum_{i \in I} d_i \equiv \frac{d_1 + d_2 + \dots + d_n + n}{2} \pmod{n}.$$

Let $A_i = \{d_i, 0\}$. Applying Theorem 2.1 inductively, we see that

$$\#(A_1 + \dots + A_n) \geq \min \left\{ n, \sum \#A_i - (n - 1) \right\} = n,$$

concluding the proof. \square

The last ingredient is the following theorem by Hall.

Theorem 2.3 ([2]). *Let A be an abelian group of order n and a_1, \dots, a_n be a numbering of the elements of A . Let $d_1, \dots, d_n \in A$ be elements such that $d_1 + \dots + d_n = 0$. Then there are permutations $\sigma, \tau \in S_n$ such that*

$$a_i - a_{\sigma(i)} = d_{\tau(i)}$$

Theorem 2.4. *Let $N = 2n$ be a positive integer. Suppose we are given n elements $d_1, d_2, \dots, d_n \in (\mathbb{Z}/N)^*$. Then there exists a partition of \mathbb{Z}/N into pairs with differences d_1, d_2, \dots, d_n .*

Proof. First from Theorem 2.2, we may assume that $d_1 + \dots + d_n \equiv n \pmod{2n}$. Now it is enough to find a numbering a_1, \dots, a_n of the odd numbers in \mathbb{Z}/N and $\sigma \in S_n$ such that $2i - a_i \equiv d_{\sigma(i)} \pmod{2n}$ for every $i = 1, \dots, n$, for then the partition in pairs $\{2, a_1\}, \{4, a_2\}, \dots, \{2n, a_n\}$ works.

Equivalently, we need to find a numbering b_1, \dots, b_n of the even numbers in \mathbb{Z}/N such that $2i - b_i \equiv d_{\sigma(i)} + 1 \pmod{N}$ for some $\sigma \in S_n$. Now since $d_i + 1$ is even for all i , this is the same as finding a permutation c_1, \dots, c_n of $\{1, \dots, n\}$ such that $i - c_i \equiv \frac{d_{\sigma(i)} + 1}{2} \pmod{n}$, for some $\sigma \in S_n$.

If we verify that $\frac{d_1 + 1}{2} + \dots + \frac{d_n + 1}{2} \equiv 0 \pmod{n}$ this will follow from Theorem 2.3. But this holds, since $d_1 + \dots + d_n \equiv n \pmod{2n}$ and therefore $(d_1 + 1) + \dots + (d_n + 1) \equiv 0 \pmod{2n}$, proving that $\frac{d_1 + 1}{2} + \dots + \frac{d_n + 1}{2} \equiv 0 \pmod{n}$. \square

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